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SYNTHESIS OF RIGID-FOLDABLE CYLINDRICAL POLYHEDRA

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Abstract: *Rigid-foldable structures are foldable surfaces consisting of rigid panels and hinges, thus can be used for wide variety of deployable structures without relying on flexible materials. In this paper, we present a family of rigid-foldable collapsible cylindrical polyhedra which is of great interest of structural engineering field. The symmetry operations in order to synthesize the cylindrical structures and their space filling tessellation are shown.*

1 INTRODUCTION

One of the basic problem areas on foldable structure is to fold a cylindrical shell in axial direction to the extreme, i.e., to a flat state, while keeping its axis and internal envelope like a bellows. There are lots of need for such structures in industry, such as deployable structures, bellows, and packaging. The present authors are specifically interested in the "bellows-type" foldable structures with a purely geometric folding mechanism while keeping substantially its sectional area during the folding/deploying maneuver.

The best way to understand the background of this problem is to consider the case of post buckling behavior of a thin cylindrical shell under axial loading. The resulting geometric form is called Yoshimura-pattern, which consists of repetition of the diamond pattern. Because the Yoshimura-pattern is obtained through the inextensional deformation process, it can be considered as the candidate of a foldable cylindrical shell. However, the matter is not simple as expected for this case. For a particular dimension of cylindrical shell, one can define the folded state or the pattern to realize a given axial length. However, once this geometric parameter is defined, it becomes a stable structure that is no more foldable without elastic deformations (Fig. 1). In short, the Yoshimura-pattern is a folded cylinder but not a continuously foldable one.

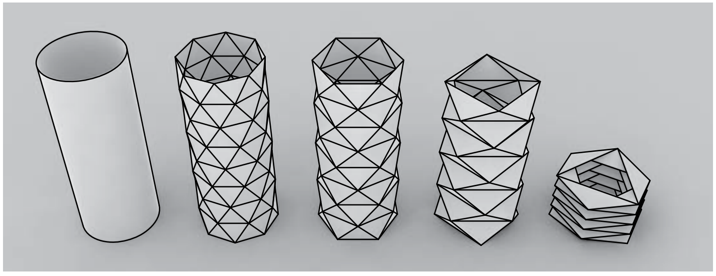


Fig.1: Yoshimura Patterns. From Left to Right: Original cylinder and Yoshimura Patterns of radial frequency of 8, 6, 4, and 3. Note that each state is static.

Though the Yoshimura-pattern provides a negative example for the present problem, the "buckling" and "concave polyhedral surface" involved in it are still the major keywords for the following researches. The typical approach on these studies depends on the model representing a cylinder with a polyhedron consisting of number of polygonal plates members. The most difficult problem is that, in the course of folding and deploying process, some amount of in plane deformation of polygonal plates are inevitable for geometric compatibility. For example, Guest and Pellegrino (1994) proposed a variation of cylindrical foldable shell by twisting Yoshimura-pattern to enable a valid state in three-dimensional and also in the flat-folded state. However, the produced designs are multi-stable structures that cannot be built from rigid or thick material. Therefore, the existence of such in-plane strains severely limits the design area of the foldable structures.

Here, we consider rigid-foldable geometry, where rigid-folding implies a continuous folding motion of a polyhedral surface in which its facets are kept congruent. Such structures can be used for wide variety of deployable structures built from rigid panels and hinges without in-plane strains. As it is already shown, not all foldable structures are rigid foldable. In this respect, a solution of bi-axial rigid folding of a flat plate exists and is known as Miura-ori (Miura, 1980) (Fig. 2). However, this principle has not successfully been applied to the case of cylinders until recently, i.e., existing designs of foldable cylindrical structures have relied on the in-plane distortion of panels or the transition of foldlines.

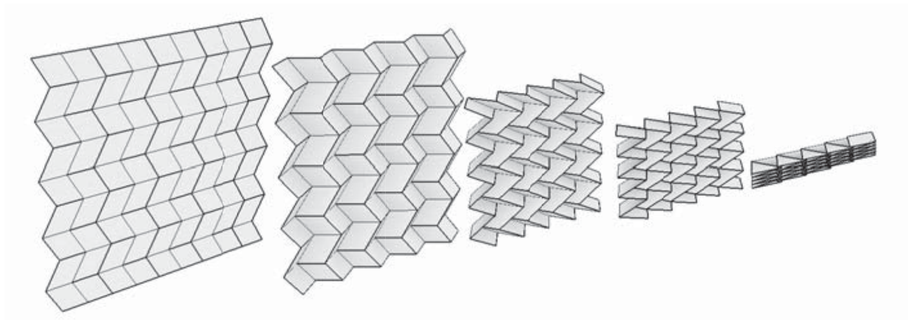


Fig.2: Miura-ori is a rigid-foldable disk.

Recently, the second author (Tachi, 2009) disclosed a novel concept of rigid-foldable polyhedral cylindrical structures. The core of the concept is that the basic unit structure is constructed by joining two pieces: a single vertex origami with four congruent parallelograms and its mirror image. In other words, the Miura-ori and its mirror image are joined. The generalization of the concept results in groups of rigid-foldable cylindrical structures.

During further studies on one of Tachi's tube structure designs, we have accidentally discovered the characteristic “star-polyhedron” embedded in it. It turned out that, the polyhedron constitutes the very core of the design (Fig. 3). In this paper, we will disclose the characteristic of the star-shaped cylindrical polyhedron by producing it by means of symmetry operations; the polyhedron can be constructed from pasting together Miura-ori vertices. Through this study, we will also show that the polyhedron is actually a space filling polyhedron. This leads to a 3D cellular structure that can simultaneously folds in x, y, and z dimensions.

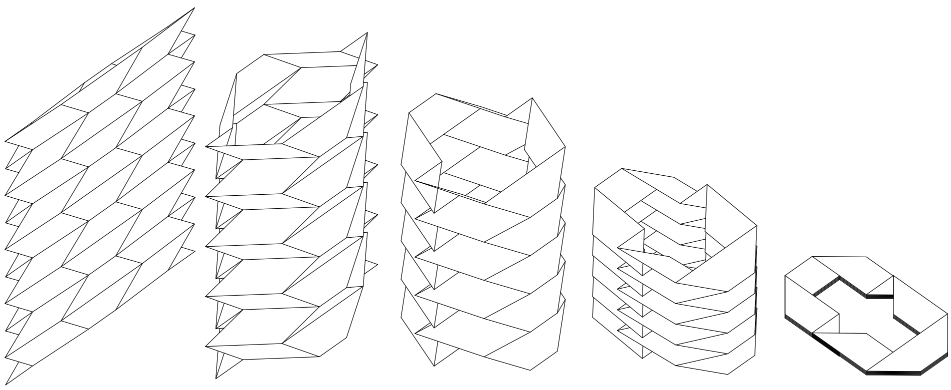
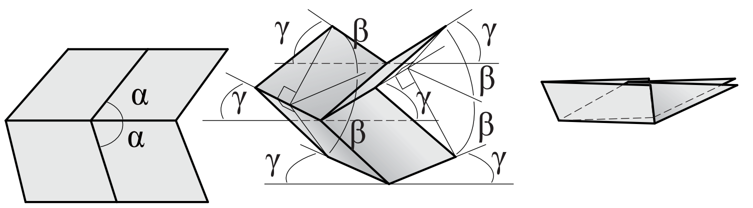


Fig.3: Star-polyhedron with a synchronized motion.

2 MODULAR STRUTCTURE

First of all, let us examine the basic geometric properties of Miura-ori and its vertex. A vertex of Miura-ori (composed of 4 foldlines) exhibits a synchronized motion of folding angles as shown in Fig. 4. This results in a one-DOF mechanism from unfolded to flat-folded states. We can repeat this vertex structure to construct an array, i.e., Miura-ori. Within the motion, the parallel lines are kept parallel, and planar polylines are kept planar. This is the very core of our study.



$$\tan\alpha \cos\beta = \tan\gamma$$

α = constant: an inner angle of parallelogram
β = variable: 1/2 of dihedral angle
γ = variable: angle between the straight foldline and the base plane

Fig. 4: A folding motion of Miura-ori vertex.

Then we consider a modular rectangle composed by 9 facets with 4 identical Miura-ori vertices as denoted in Fig. 5. Because of the repeating structure, this surface transforms preserving the parallelism of foldlines and the planarity and congruency of end polylines. This enables us to set a local coordinate that works throughout the transformation (Fig. 6). Specifically, we extract a set of parallel lines $\{A_0B_0, ..., A_3B_3, C_0D_0, ..., C_3D_3\}$ and use this direction as the x-axis. Symmetric structure of Miura-ori vertex forces polyline $A_iB_iC_iD_i$ (i = 0, 1, 2, 3) to be planar; here we term the containing plane i-plane. We define xy-plane as 0-plane. Since we cut parallel lines by perpendicular boundary polylines $A_0A_1A_2A_3$ and $D_0D_1D_2D_3$ (termed A- and D-polylines, repsectively), each of the folded state of polylines lies on a plane perpendicular to x-axis, which is termed A- or D-plane. Here, note that A- and D-polylines are congruent to each other and each has 180° rotational symmetry around the axis through the midpoint of A_1A_2 or D_1D_2 along x-direction. We term these axes A- and D-axes. The module transforms so that the surface boundary is tangent to A-, D-, 0-, and 3-planes, where A- and D-planes are kept parallel to yz-plane, and 0- and 3-planes to xy-plane.

In a design context, we can generalize the rectangular module so that it has 5, 7, or $2n$+1 columns by adding extra 2, 4, or $2n$ zigzagged foldlines similar to B- and C-polylines. This is because the property of the boundary (e.g., A- and D-planes) in this case is equivalent to that in the 3 columns case. Hence our method described in this paper is essentially a general approach to merge 4 pieces of Miura-ori to form a cylindrical surface.

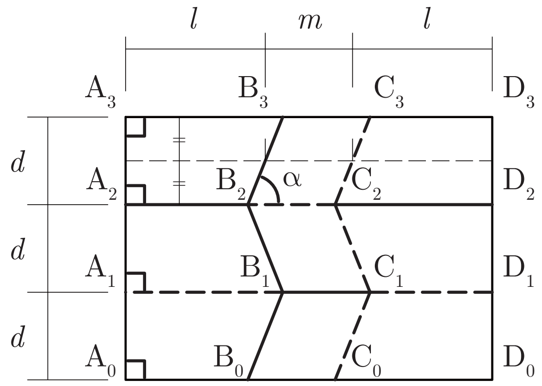


Fig 5: Notation of the module.

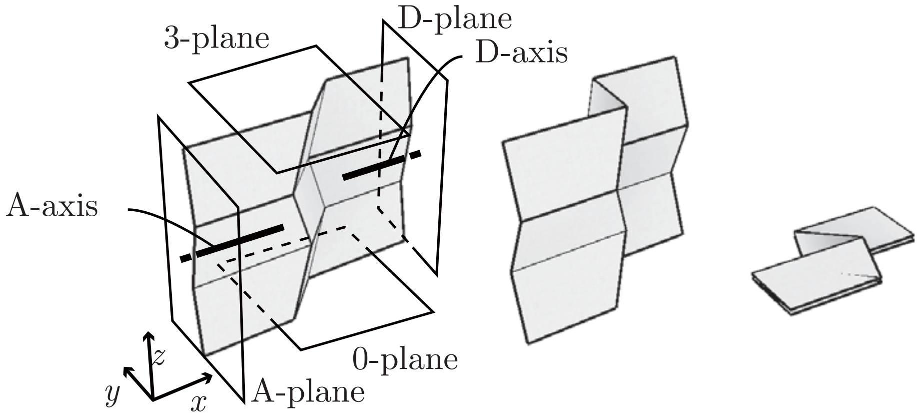


Fig 6: Coordinate system.

3 CYLINDRICAL POLYHEDRA

Described co-planarity and congruency of edges enable us to paste modules edge-to-edge with symmetric operation without producing any gap between them throughout the transformation. The following shows the process (Fig. 7 and Fig.8).

1) Produce a mirror image of the module with respect to its D plane. This connects two units (in any folded state) without gap. In this procedure, the A-axes of the original and mirrored modules coincide since the x-directions of the original and copied modules are opposite (Fig.7 middle).

2) With respect to the common A-axis, rotate the connected modules by 180°. This also validly connects four modules in any folded state since A-polyline on the boundary has rotational symmetry with respect to the axis. Here, line A_iB_i is rotated to the position of $A_{3-i}B_{3-i}$ (Fig.7 right).

3) The rotation makes the connected part doubly covered by facets (Fig. 8(a)). However the difference between the lengths of A_iB_i and $A_{3-i}B_{3-i}$ produces the mismatch. If we remove the doubly covered part, we obtain the singly covered star-shaped polyhedron as shown in Fig. 8 (b).

4) Since 0- and 3-planes are kept planar by the process 1) and 2), we can also extend this geometry to the *z*-axial direction by mirror reflecting with respect to 3-plane. We can also cut out one or two of the three-rows of the module in order to adjust the corrugation frequency (Fig. 8 (c)).

Once the cylindrical polyhedron is built, we can naturally extract a two-row module and its glide reflectance (Fig. 8 (d)). In fact, the rotation in 2) is essentially equivalent to the glide reflection with respect to a plane through A-axis and parallel to *zx*-plane with the half wavelength translation in z-direction.

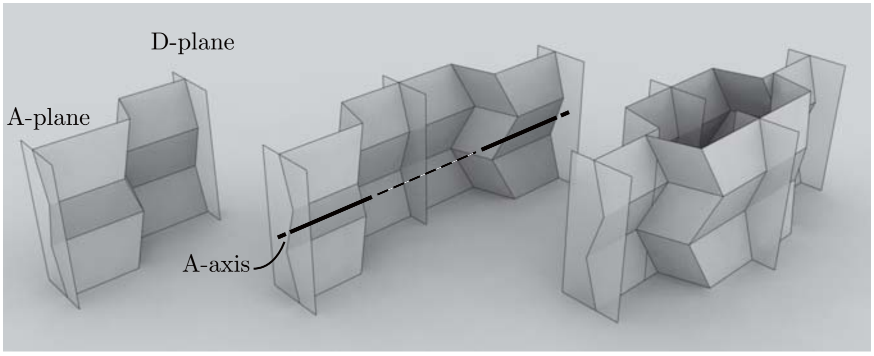


Fig 7: Symmetry operation to construct a cylinder by composing the modules.

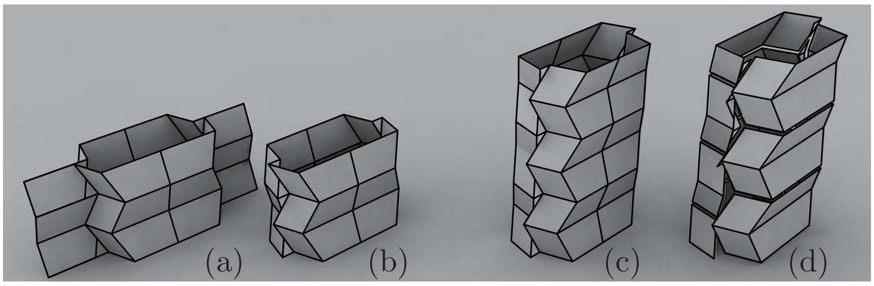


Fig. 8: Trimming out the singly covered star-shaped polyhedron and extending the cylinder by repeating mirror reflection to *z*-direction.

The pattern of the cylinder can be parameterized by the dimension of the module represented by α, *l*, *m*, and *d* and the number of repetition *N*. On the other hand, the transformation of the shape is represented by single variable β (0≤ β ≤90°) representing the half of dihedral angle of any edge between adjacent facets in the same column such as A_1B_1 or B_2C_2 (all of them are equal). Patterns with different α produce different folding motions. Fig. 9, 10 and 11 show the transformation of cylinders in case of α=30°, 45°, and 54°, respectively, using common *l*=1, *m*=2, *d*=1, *N*=9.

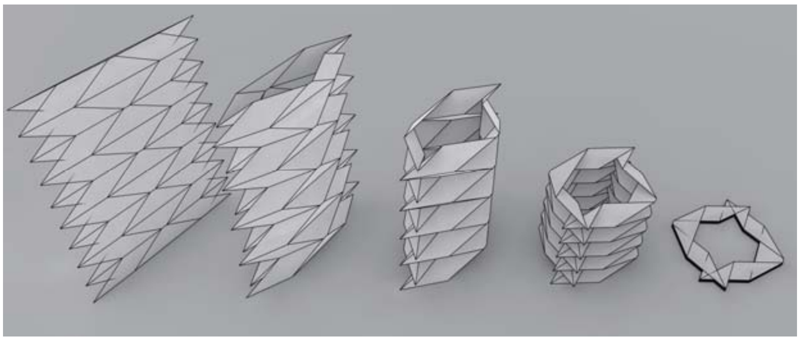


Fig. 9: The cylinder of α=30° at β = 89°, 67.5°, 45°, 22.5°, 1° (from left to right).

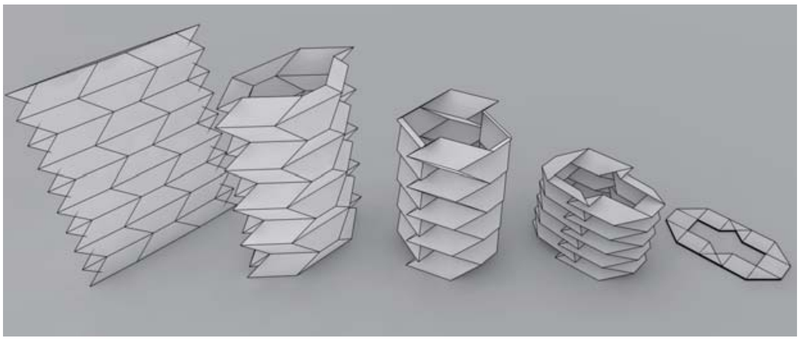


Fig. 10: The cylinder of α=45° at β = 89°, 67.5°, 45°, 22.5°, 1° (from left to right).

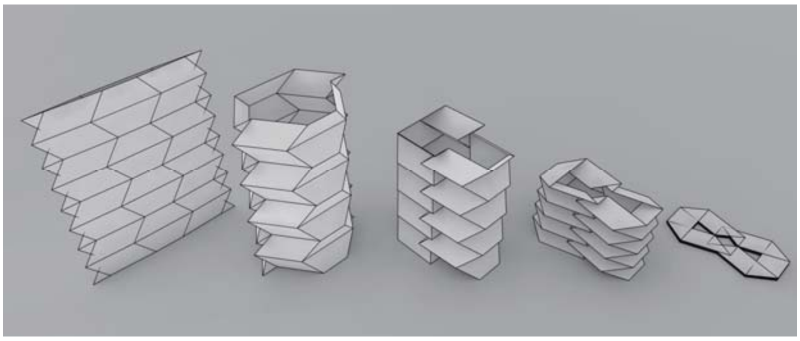


Fig. 11: The cylinder of α=54° at β = 89°, 67.5°, 45°, 22.5°, 1° (from left to right).

Because the proposed cylindrical polyhedra are composed of rigid facets, we can attach solid panels to the facets while producing kinetic motion without distorting them. By applying the thickening method introduce by (Tachi, 2010), we could obtain a thick panels structure that follows the kinetic motion of proposed cylindrical polyhedron as shown in Fig. 12. This indicates that our structure can be utilized in different scales and design contexts.

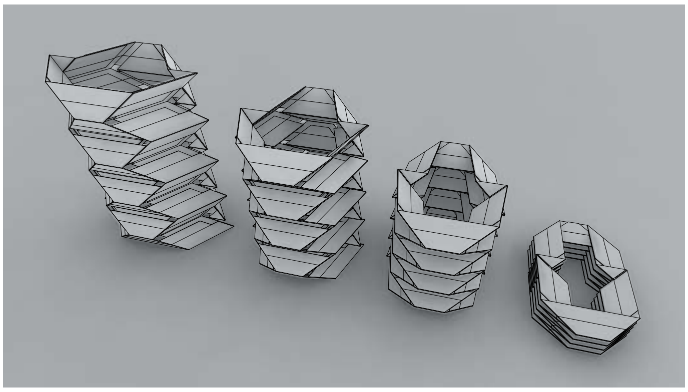


Fig. 12: Cylindrical polyhedra using thick panels. (α=45°)

4 SPACE FILLING AND 3D CELLULAR STRUCTURE

The proposed cylinders can periodically tessellate the three-dimensional space without producing any gap as shown in Fig. 13. In other words, the cylinders consists a 3D cellular structure. This tessellation property can be preserved for any folded state; hence this tessellation has a synchronized one-DOF rigid-folding motion inherent from the cylinder. This volumetric polyhedral complex can smoothly fold itself to two flat states. The validity of the structure can be similarly described by applying symmetry operations to the modules we used for synthesizing cylinders.

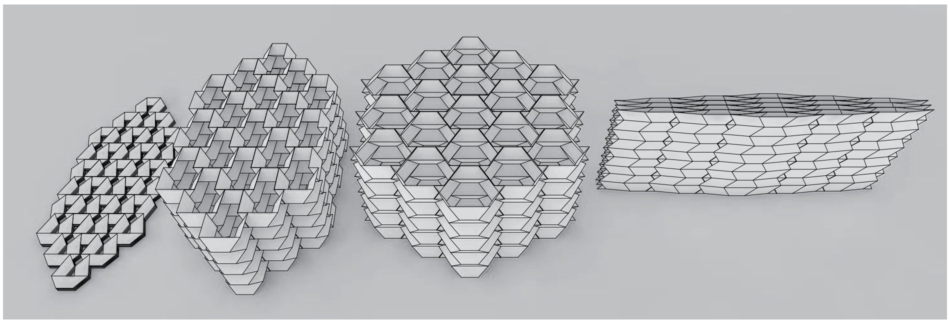


Fig. 13: 3D cellular structure in folding motion.

In the construction of a cylinder, we used D-plane for the reflection and A-axis for the rotation. However, because of the symmetric structure of the module, A and B can substitute C and D, and vice versa via 180° rotation with respect to the axis through the center and along the z-axis. Therefore, a pair of cylinders is produced from a single module as shown in Fig. 14 left. Constructing a cylinder produces four congruent modules. We recursively construct cylinders from already built modules. This procedure forms a regular cellular structure that fills the three-dimensional space. In order to understand this regularity, consider the contact points of A- and D- polylines with 0-plane or the xy-plane. The points are located on the vertices of rhombus whose diagonals are along x- and y-axes. Therefore constructing a cylinder from a module adds a rhombus pasted edge-to-edge to the existing ones, and thus A- and D- polylines are essentially located at the nodes of rhombus grid on xy-plane (Fig. 14 right).

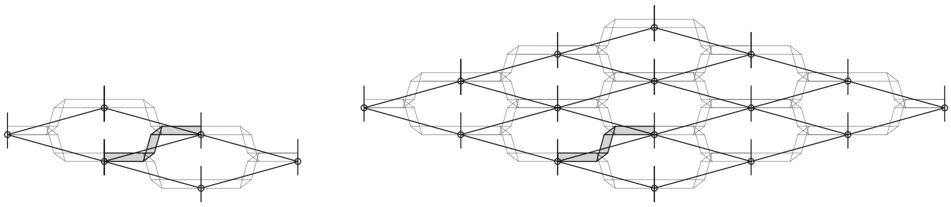


Fig. 14: Left: A pair of cylinders produced from one module.
Right: Rhombus grid on whose vertices A- and D- polylines are located.

Now we obtained a foldable cellular structure of single-walled (non-manifold) polygonal mesh. Then, we show that the star-shaped cylindrical polyhedra fill the space with every part of the surface touching an adjacent polyhedron. This means that the polyhedra can completely doubly cover the cellular polygonal mesh. We first decompose the cylinder into 6 parts, comprising of three pairs of congruent corrugated surfaces P-P′, Q-Q′, and R-R′ (Fig. 15). Then we consider 6 cases of translation, each of which moves one part of the unit to its congruent counterpart, e.g., P to P′ or Q′ to Q. Under such transformations, cylindrical unit moves to adjacent six units. Therefore every part of the cylinder is shared by and only by the corresponding adjacent cylinder as shown in Fig. 16. Here, note that the orientations of congruent counterparts (e.g., P and P′) are opposite. This makes adjacent cylinders "touch" each other. Therefore every surface is doubly covered, and thus the cylindrical polyhedron can tessellate the three-dimensional space.

5 CONCLUDING REMARKS

First, we have accidentally discovered the characteristic "star-polyhedra" following the general concept proposed by Tachi (2009). The symmetry operations in order to synthesize the cylindrical structures are shown. The family of the polyhedra can be used for wide variety of deployable structures and other mechanisms.

Second, during the detailed study on the polyhedra, we have unexpectedly discovered that it has the space filling property in spite of its star-shape. The resulting volumetric polyhedral complex can smoothly fold itself to two flat states. This novel concept will surely attract the interests of potential users.

Through this study, we were able to present a novel concept of polyhedra having unprecedented properties.

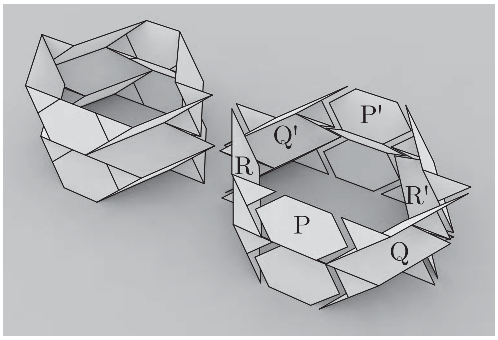


Fig. 15: Cylinder decomposed into 6 parts.

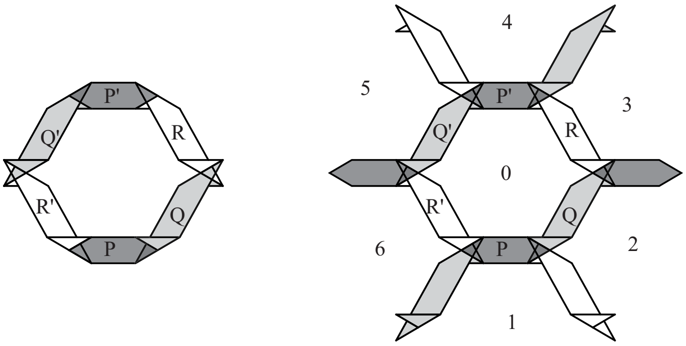


Fig. 16: Correspondence of parts to adjacent cylinders. Parts {P, P′, Q, Q′, R, R′} of cylinder 0 is shared by adjacent cylinders {1, 4, 2, 5, 3, 6}, respectively.

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